# VALUES OF MODULAR FUNCTIONS AT REAL QUADRATICS AND CONJECTURES OF KANEKO

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Abstract. In 2008, M. Kaneko made several interesting observations about the values of the modular $j$ invariant at real quadratic irrationalities. The values of modular functions at real quadratics are defined in terms of their cycle integrals along the associated geodesics. In this paper we prove some of the conjectures of M. Kaneko for a general modular function.

## 1. Introduction

Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and $j(z) = \frac{1}{q} + 744 + 196884q + \cdots$ be the classical Klein's modular invariant. The values of $j$ at imaginary quadratic irrationalities have a long and rich history going back to Kronecker and Weber and play an important role in the theory of complex muliplication. They have also seen considerable recent interest due to the beautiful results of Borcherds and Zagier, which relate their traces to the coefficients of half integral weight modular forms.

For a real quadratic irrationality $w \in \mathbb{Q}(\sqrt{D})$, the "value" $f(w)$ of a general modular function $f$ has been defined only recently in [4] and [3] using their periods along the closed geodesic associated to $w$. In [3], their traces $\mathrm{Tr}_D j := \sum j(w)$, where the sum is over the finitely many ideal classes of $\mathbb{Q}(\sqrt{D})$, were related to the coefficients of mock modular forms, generalizing the results of Borcherds and Zagier. As was conjectured in [3] and proved in [2] and [5], we also know that

$$\frac{\mathrm{Tr}_D(j)}{\mathrm{Tr}_D(1)} \to 720, \tag{1}$$

as fundamental discriminants $D \to \infty$.

In this paper we turn to the study of the individual values of modular functions at real quadratic irrationalities. Even though the arithmetic/algebraic properties of their individual values remain inaccessible, in [4], Kaneko studied the numerical values of $j(w)$ and made several remarkable observations. Some of his observations involving the values of $j$ at Markov quadratics were recently proved in [1]. For a general quadratic irrationality, among his many observations, we note the following boundedness conjecture.

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**Conjecture 1** (Kaneko)**.** *Let $w \in \mathbb{Q}(\sqrt{D})$ be a real quadratic irrationality. Then*

$$\mathrm{Re}(j^{nor}(w)) \in [j^{nor}((1+\sqrt{5})/2), 744] \quad \textit{and} \quad \mathrm{Im}(j^{nor}(w)) \in (-1,1)$$

*where*

$$j^{nor}(w) := \frac{1}{2\log\varepsilon}\int_{C_w} j(z)ds,$$

*$C_w$ is the closed geodesic associated to $w$ in $\Gamma\backslash\mathcal{H}$, $\varepsilon > 1$ is the smallest unit with positive norm in $\mathbb{Q}(\sqrt{D})$ and $ds$ is the hyperbolic arc length.*

In what follows we take a closer look at the individual values $f(w)$ for any real quadratic irrationality $w$ and any modular function $f(z)$ which takes real values on the geodesic arc $\{e^{i\theta}\colon \pi/3 \le \theta \le 2\pi/3\}$. As a special case of our results we prove

**Theorem 1.** *Let $j(z)$ be the classical modular invariant. Let $w$ be a real quadratic number and $(\overline{a_1,\ldots,a_n})$ be its period in the negative continued fraction expansion. Then we have*

*(1) For any positive integer $N > 2$, the value $j^{nor}((\overline{N}))$ for the quadratic number $w = (\overline{N})$ is real and*

$$\lim_{N\to\infty} j^{nor}((\overline{N})) = 744.$$

*(2)* $\mathrm{Re}(j^{nor}(w)) \le 744$.

*(3) If all the partial quotients $a_r$ in the period of $w$ satisfy $a_r \ge 3M$ with $M = e^{55}$ then*

$$\mathrm{Re}(j^{nor}(w)) \ge j^{nor}((1+\sqrt{5})/2).$$

The second part of Theorem 1 proves the upper bound conjectured by Kaneko for any $w$ while the first part shows that in fact this bound is optimal. Part(1), when combined with the limiting behavior of the traces in equation (1), also has the amusing corollary that there are infinitely many discriminants of the form $D = N^2 - 4$ with class number bigger than one and only finitely many such discriminants with class number one. This corollary was also observed in the master thesis of S. Päpcke. In [6] another proof of part (1) of Theorem 1 was also given.

It is worth noting that Part (3) of the theorem can be rephrased in terms of the diophantine properties of the quadratic numbers, more precisely in terms of the Lagrange spectrum. For any irrational number $x$, let $\|x\|$ denote the distance from $x$ to a closest integer. Then recall that the Lagrange spectrum $L$ is defined as

$$L := \{\nu(x)\}_{x\in\mathbb{R}} \subseteq \left[0, 1/\sqrt{5}\right] \quad \text{with} \quad \nu(x) = \liminf_{q\to\infty} q\|qx\|.$$

It is known that if $x$ has a continued fraction $(a_1, a_2, \ldots)$ then $\nu(x) \le \inf_{r\ge 1} a_r^{-1}$. Hence part (3) of Theorem 1 proves the lower bound conjectured by Kaneko for the quadratic irrationalities $w$ with $\nu(w) \in [0, 1/3M]$.

After giving some preliminaries in the next section, we will collect several technical results in Section 3 that will be needed in the sequel. In Section 4, we start by proving the first part of Theorem 1 for a general modular function which takes real values on the geodesic arc $\{e^{i\theta} : \pi/3 \leq \theta \leq 2\pi/3\}$. The results from Section 3 are then used to prove Theorem 4.2, which compares the values of modular functions at different quadratic irrationalities by comparing their corresponding partial quotients. Theorem 4.2 is the main theorem of this paper and the results (2) and (3) in Theorem 1 follow as its simple corollaries.

## 2. PRELIMINARIES

Let $w$ be a real quadratic irrationality and $\tilde{w}$ be its conjugate. $w$ and $\tilde{w}$ are the roots of a quadratic equation

$$ax^2 + bx + c = 0 \qquad (a, b, c \in \mathbb{Z}, \quad (a, b, c) = 1)$$

with discriminant $D = b^2 - 4ac > 0$. We choose $a, b, c$ such that $w = \frac{-b+\sqrt{D}}{2a}$, $\tilde{w} = \frac{-b-\sqrt{D}}{2a}$. The geodesic $S_w$ in $\mathcal{H}$ joining $w$ and $\tilde{w}$ is given by the equation

$$a|z|^2 + b\,\mathrm{Re}(z) + c = 0 \qquad (z \in \mathcal{H}).$$

The stabilizer $\Gamma_w$ of $w$ in $\Gamma$ preserves the quadratic form $Q_w = [a, b, c]$, and hence $S_w$. Let $A_w$ be the generator of the infinite cyclic group $\Gamma_w$ with

$$A_w = \begin{pmatrix} \frac{1}{2}(t - bu) & -cu \\ au & \frac{1}{2}(t + bu) \end{pmatrix},$$

where $(t, u)$ is the smallest positive solution to Pell's equation $t^2 - Du^2 = 4$. We denote by $\varepsilon = (t + u\sqrt{D})/2$ the smallest unit with positive norm, which is either a fundamental unit or the square of a fundamental unit.

For any modular function $f$, since the group $\Gamma_w$ preserves the expression $f(z)Q_w(z, 1)^{-1}dz$, one can define the cycle integral of $f$ along $C_w = \Gamma_w \backslash S_w$, also viewed as the "value" of $f$ at $w$, by the complex number

$$f(w) := \int_{C_w} \frac{\sqrt{D} f(z)}{Q_w(z, 1)} dz.$$

The factor $\sqrt{D}$ is introduced here for convenience. The integral defining $f(w)$ is $\Gamma$-invariant and can in fact be taken along any path in $\mathcal{H}$ from $z_0$ to $A_w^{-1} z_0$, where $z_0$ is any point in $\mathcal{H}$. Note that this gives an orientation on $S_w$ from $w$ to $\tilde{w}$, which is counterclockwise if $a > 0$ and clockwise if $a < 0$. We normalize the number $f(w)$ by the length of the geodesic $C_w$ which is given by

$$\int_{C_w} \frac{\sqrt{D}}{Q_w(z, 1)} dz = 2 \log \varepsilon$$

and we define the normalized value as

$$f^{nor}(w) := \frac{f(w)}{2 \log \varepsilon}.$$

For any quadratic irrationality $w$, it is known that the hyperbolic element $A_w$ is conjugate to a word in positive powers of $T$ and $V$, where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For a quadratic number $w$ which has a purely periodic negative continued fraction, such a word is the element $A = A_1^{-1} \dots A_{\ell_w}^{-1}$ that fixes $w-1$. Here $A_i \in \{T^{-1}, V^{-1}\}$ for $1 \le i \le \ell_w$ are given by the finite algorithm:

$$w^0 = w - 1, \qquad w^{i+1} = A_{i+1}(w^i) \qquad (i \ge 0), \tag{2}$$

where

$$A_{i+1} = \begin{cases} T^{-1} & \text{if } \lfloor w^i \rfloor \ge 1, \\ V^{-1} & \text{otherwise.} \end{cases}$$

Since the cycle integral defining $f(w)$ is a class invariant, we have the following simple lemma.

**Lemma 2.1.** *For a real quadratic $w$ we have*

$$f(w) = f(w-1) = \int_\rho^{\rho^2} \sum_{i=0}^{\ell_w - 1} f(z) \left( \frac{1}{z - w^i} - \frac{1}{z - \tilde{w}^i} \right) dz \tag{3}$$

*where $\rho = e^{\pi i/3}$.*

Let $w$ be a purely periodic quadratic number and $\overline{a_1, \dots, a_n}$ be its period in its negative continued fraction. We let

$$\begin{aligned} w_{r,k} &= (k, \overline{a_{r+1}, \dots, a_n, a_1, \dots, a_r}), \\ \tilde{w}_{r,k} &= -(a_r - k, \overline{a_{r-1}, \dots, a_1, a_n, \dots, a_r}). \end{aligned}$$

Note that each $w_{r,k}$ arises for $1 \le k \le a_r - 1$ as one of the $w^i$'s from (2).

If $r$ is fixed and there is no possible confusion, we will drop the dependence on $r$ and write $w_k := w_{r,k}$ for simplicity.

## 3. Technical lemmas

Let $w = (\overline{a_1, \dots, a_n})$ and $v = (\overline{b_1, \dots, b_m})$ be two purely periodic quadratic numbers with $n \ge m$. If $n > m$, then we define $b_{m+1}, b_{m+2}, \dots, b_n$ by cycling back to $b_1, b_2, etc.$ For a fixed $r$, let

$$S_{w,v}(z, r) := \sum_{k=1}^{a_r - 1} \left( \frac{1}{z - w_{r,k}} - \frac{1}{z - \tilde{w}_{r,k}} \right) - \sum_{k=1}^{b_r - 1} \left( \frac{1}{z - v_{r,k}} - \frac{1}{z - \tilde{v}_{r,k}} \right). \tag{4}$$

**Theorem 3.1.** *Let $w = (\overline{a_1, \dots, a_n})$ and $v = (\overline{b_1, \dots, b_m})$ be two purely periodic quadratic numbers. If $a_r \le b_r$ for all $r = 1, \dots, n$, then we have for $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$*

$$C_2(r) < \mathrm{Re}(S_{w,v}(e^{i\theta}, r)) < C_1(r) \tag{5}$$

*with*

$$C_1(r) = 12.6 + \frac{7}{3}\Big( \log\Big(\frac{b_r - 3 + k_1}{a_r - 3 + k_1}\Big) - \frac{4}{a_r - 3 + k_1} - \frac{3}{(a_r - 3 + k_1)^2}\Big),$$

$$C_2(r) = -3.925 + \Big(1 - \frac{2}{a_r} + \frac{1}{2a_r^2}\Big)\Big(2\log\Big(\frac{b_r + 1 + k_0}{a_r + 2 + k_0}\Big) - \frac{5}{a_r + 2 + k_0}\Big),$$

*and*

$$-13.01 < \mathrm{Im}(S_{w,v}(e^{i\theta}, r)) < 14.99. \tag{6}$$

To prove Theorem 3.1, we start with the following two simple lemmas.

For $x, y \in \mathbb{R}$, $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, let

$$(x-y)F_{x,y}(\theta) = \mathrm{Re}\Big(\frac{1}{e^{i\theta} - x} - \frac{1}{e^{i\theta} - y}\Big)$$

and

$$(x-y)G_{x,y}(\theta) = \mathrm{Im}\Big(\frac{1}{e^{i\theta} - x} - \frac{1}{e^{i\theta} - y}\Big),$$

so that

$$F_{x,y}(\theta) = \frac{\cos^2\theta - (x+y)\cos\theta + xy - \sin^2\theta}{((\cos\theta - x)^2 + \sin^2\theta)((\cos\theta - y)^2 + \sin^2\theta)}$$

and

$$G_{x,y}(\theta) = \frac{\sin\theta(x + y - 2\cos\theta)}{((\cos\theta - x)^2 + \sin^2\theta)((\cos\theta - y)^2 + \sin^2\theta)}.$$

**Lemma 3.2.** *As a function of $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, $F_{x,y}(\theta)$ satisfies the following properties:*

*(i) $F_{x,y}(\theta)$ is decreasing for $x, y \in (2, \infty)$ and increasing for $x, y \in (-\infty, -2)$.*

*(ii) For $0 \le |x| \le 1$ and $0 \le |y| \le 1$, we have*

$$-1.4 < F_{x,y}(\theta) < 0.2.$$

*(iii) For $1 \le |x| \le 2$ and $1 \le |y| \le 2$, we have*

$$-0.5 < F_{x,y}(\theta) < 0.2.$$

*(iv) For $x \ge 2$ and $y \le -2$,*

$$\frac{\min(xy - \frac{|x+y|}{2} - \frac{1}{2}, xy - 1)}{((-\frac{1}{2} + x)^2 + \frac{3}{4})((\frac{1}{2} + y)^2 + \frac{3}{4})} < F_{x,y}(\theta) < \frac{xy + \frac{|x+y|}{2} - \frac{1}{2}}{((\frac{1}{2} + x)^2 + \frac{3}{4})((-\frac{1}{2} + y)^2 + \frac{3}{4})}.$$

*Proof.* The last assertion is straightforward using the definition of the function $F_{x,y}(\theta)$. The other three assertions can be easily verified numerically. For example, for $x > 2$ and $y > 2$ the maximum of the derivative $\frac{dF}{d\theta}$ is $-0.00504971$ where as its minimum is $-0.19245$. This shows that for $x > 2$ and $y > 2$, $\frac{dF}{d\theta} < 0$ and hence $F_{x,y}(\theta)$ is decreasing.

□

Similarly one can prove

**Lemma 3.3.** *As a function of $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, $G_{x,y}(\theta)$ satisfies the following properties:*

*(i) $G_{x,y}(\theta)$ is decreasing for $x, y \in (1, \infty)$ and for $x, y \in (-\infty, -1)$.*
*(ii) $G_{x,y}(\theta)$ is increasing for $x \in (1, \infty)$ and $y \in (-\infty, -1)$.*
*(iii) For $0 \leq |x| \leq 1$ and $0 \leq |y| \leq 1$, we have*

$$-0.9 \approx G_{0,0}(e^{\pi i/3}) \leq G_{x,y}(\theta) \leq G_{0,0}(e^{2\pi i/3}) \approx 0.9.$$

*Proof. (Theorem 3.1)*: We start by grouping some of the terms from $S_{w,v}$ into two sums $S_1$ and $S_2$ ($S_1$ corresponding to terms $w_k, v_k$ and $S_2$ to conjugates $\tilde{w}_k, \tilde{v}_k$) whose contribution, as we will see, will be minor. Since $r$ is fixed in the whole proof, we drop the dependence on $r$ in the notation for all these sums. Define

$$S_1(z) := \sum_{k=1}^{a_r-1} \frac{1}{z - w_k} - \frac{1}{z - v_k},$$

$$S_2(z) := \sum_{k=1}^{a_r-1} \frac{1}{z - \tilde{w}_k} - \frac{1}{z - \tilde{v}_{b_r - a_r + k}}.$$

We group the remaining terms from $S_{w,v}$ in the sum $S_3$, which will be the major contribution:

$$S_3(z) = \sum_{k=a_r}^{b_r-1} \frac{1}{z - v_k} - \frac{1}{z - \tilde{v}_{b_r - k}}.$$

Hence,

$$S_{w,v}(z, r) = S_1(z) - S_2(z) - S_3(z). \tag{7}$$

3.1. **Proof of equation** (5)**.** The real part of $S_1(e^{i\theta})$ satisfies

$$\mathrm{Re}(S_1(e^{i\theta})) = (w_0 - v_0) \sum_{k=1}^{a_r-1} F_{w_k, v_k}(\theta).$$

For $3 \leq k \leq a_r - 1$, since $\lfloor w_k \rfloor = \lfloor v_k \rfloor = k - 1 > 2$, by Lemma 3.2 (i),

$$F_{w_k,v_k}(\theta) < F_{w_k,v_k}(\pi/3) < \frac{k^2 - k + \frac{1}{2}}{(k^2 - 3k + 3)^2}$$

and

$$F_{w_k,v_k}(\theta) > F_{w_k,v_k}(2\pi/3) > \frac{k^2 - k - \frac{1}{2}}{(k^2 + k + 1)^2}.$$

Since $-1 < w_0 - v_0 < 0$ and using Lemma 3.2 (ii)-(iii), we have

$$\begin{aligned} \mathrm{Re}(S_1(e^{i\theta})) &< (w_0 - v_0)\Big( -1.9 + \sum_{k=3}^{a_r-1} \frac{k^2 - k - \frac{1}{2}}{(k^2 + k + 1)^2}\Big) \\ &< (w_0 - v_0)(-1.9 + 0.2162) < 1.683 \end{aligned} \tag{8}$$

and

$$\operatorname{Re}(S_1(e^{i\theta})) > -0.4 - \sum_{k=3}^{a_r-1} \frac{k^2-k+\frac{1}{2}}{(k^2-3k+3)^2}$$

$$> -0.4 - 1.26271 > -1.66271. \tag{9}$$

The real part of $S_2(e^{i\theta})$ satisfies

$$\operatorname{Re}(S_2(e^{i\theta})) = (\tilde{w}_{a_r} - \tilde{v}_{b_r}) \sum_{k=1}^{a_r-1} F_{\tilde{w}_k,\tilde{v}_{b_r-a_r+k}}(\theta).$$

For $1 \le k \le a_r - 3$, $\lfloor \tilde{w}_k \rfloor = \lfloor \tilde{v}_{b_r-a_r+k} \rfloor = -a_r + k < -2$ and hence by Lemma 3.2 (i), we have

$$F_{\tilde{w}_k,\tilde{v}_{b_r-a_r+k}}(\theta) < F_{\tilde{w}_k,\tilde{v}_{b_r-a_r+k}}(2\pi/3) < \frac{(a_r-k)^2-(a_r-k)+\frac{1}{2}}{((a_r-k)^2-3(a_r-k)+3)^2}$$

and

$$F_{\tilde{w}_k,\tilde{v}_{b_r-a_r+k}}(\theta) > F_{\tilde{w}_k,\tilde{v}_{b_r-a_r+k}}(\pi/3) > \frac{(a_r-k)^2-(a_r-k)-\frac{1}{2}}{((a_r-k)^2+(a_r-k)+1)^2}.$$

Since $\lfloor \tilde{w}_{a_r} \rfloor = \lfloor \tilde{v}_{b_r} \rfloor = 0$ and $b_r \ge a_r$, $0 < \tilde{w}_{a_r} - \tilde{v}_{b_r} < 1$ and using Lemma 3.2 (ii)-(iii),

$$\operatorname{Re}(S_2(e^{i\theta})) < 0.4 + \sum_{k=1}^{a_r-3} \frac{(a_r-k)^2-(a_r-k)+\frac{1}{2}}{((a_r-k)^2-3(a_r-k)+3)^2}$$

$$< 0.4 + \sum_{k=3}^{a_r-1} \frac{k^2-k+\frac{1}{2}}{(k^2-3k+3)^2}$$

$$< 0.4 + 1.26271 < 1.66271 \tag{10}$$

and

$$\operatorname{Re}(S_2(e^{i\theta})) > (\tilde{w}_{a_r} - \tilde{v}_{b_r}) \Big( -1.9 + \sum_{k=1}^{a_r-3} \frac{(a_r-k)^2-(a_r-k)-\frac{1}{2}}{((a_r-k)^2+(a_r-k)+1)^2} \Big)$$

$$> -1.9. \tag{11}$$

The real part of $S_3(e^{i\theta})$ satisfies

$$\operatorname{Re}(S_3(e^{i\theta})) = \sum_{k=a_r}^{b_r-1} (2k + v_0 - \tilde{v}_{b_r}) F_{v_k,\tilde{v}_{b_r-k}}(\theta).$$

Since $-\lfloor \tilde{v}_{b_r-k} \rfloor = \lfloor v_k \rfloor = k$, for $k \ge 3$, by Lemma 3.2 (iv),

$$\frac{-k^2-\frac{3}{2}}{(k^2-3k+3)^2} < F_{v_k,\tilde{v}_{b_r-k}}(\theta) < \frac{-k^2+2k-\frac{1}{2}}{(k^2+k+1)^2}.$$

Therefore,

$$\begin{aligned}
\mathrm{Re}(S_3(e^{i\theta})) &< 0.6 + \sum_{k=a_r+k_0}^{b_r-1+k_0} (2k-1)\frac{-k^2+2k-\frac{1}{2}}{(k^2+k+1)^2} \\
&< 0.6 - \sum_{k=a_r+2+k_0}^{b_r+1+k_0} (2k-5)\frac{1-\frac{2}{a_r}+\frac{1}{a_r^2}}{k^2} \\
&< 0.6 - \Big(1-\frac{2}{a_r}+\frac{1}{2a_r^2}\Big)\Big(2\log\Big(\frac{b_r+1+k_0}{a_r+2+k_0}\Big) - \frac{5}{a_r+2+k_0}\Big) \qquad (12)
\end{aligned}$$

with $k_0 = 1$ if $a_r = 2$ or $k_0 = 0$ otherwise. We also have

$$\begin{aligned}
\mathrm{Re}(S_3(e^{i\theta})) &> -2 + \sum_{k=a_r+k_1}^{b_r-1+k_1} \frac{2k(-k^2-\frac{3}{2})}{(k^2-3k+3)^2} \\
&> -9 - \frac{7}{3}\sum_{k=a_r-3+k_1}^{b_r-3+k_1} \frac{k+3}{k^2} \\
&> -9 - \frac{7}{3}\Big(\log\Big(\frac{b_r-3+k_1}{a_r-3+k_1}\Big) - \frac{4}{a_r-3+k_1} - \frac{3}{(a_r-3+k_1)^2}\Big) \qquad (13)
\end{aligned}$$

with

$$k_1 = \begin{cases} 2 & \text{if } a_r = 2 \\ 1 & \text{if } a_r = 3 \\ 0 & \text{if } a_r \geq 4. \end{cases}$$

By (7), (8), (11) and (13) we have that

$$\mathrm{Re}(S_{w,v}(e^{i\theta}), r) < 12.6 + \frac{7}{3}\Big(\log\Big(\frac{b_r-3+k_1}{a_r-3+k_1}\Big) - \frac{4}{a_r-3+k_1} - \frac{3}{(a_r-3+k_1)^2}\Big) \qquad (14)$$

and by (9), (10) and (12),

$$\mathrm{Re}(S_{w,v}(e^{i\theta}), r) > -3.925 + \Big(1-\frac{2}{a_r}+\frac{1}{2a_r^2}\Big)\Big(2\log\Big(\frac{b_r+1+k_0}{a_r+2+k_0}\Big) - \frac{5}{a_r+2+k_0}\Big) \qquad (15)$$

3.2. **Proof of equation** (6)**.** The imaginary part of $S_1(e^{i\theta})$ satisfies

$$\mathrm{Im}(S_1(e^{i\theta})) = (w_0 - v_0)\sum_{k=1}^{a_r-1} G_{w_k,v_k}(\theta).$$

For $k \geq 2$, by Lemma 3.3 (i),

$$G_{w_k,v_k}(\theta) < G_{w_k,v_k}(e^{\pi i/3}) < \frac{\frac{\sqrt{3}}{2}(2k-1)}{(k^2-3k+3)^2}$$

and

$$G_{w_k,v_k}(\theta) > G_{w_k,v_k}(e^{2\pi i/3}) > \frac{\frac{\sqrt{3}}{2}(2k-1)}{(k^2+k+1)}.$$

Since $-1 < w_0 - v_0 < 0$ and using Lemma 3.3 (iii), we have

$$\begin{aligned} \operatorname{Im}(S_1(e^{i\theta})) &< (w_0 - v_0)\Big( -0.9 + \sum_{k=2}^{a_r-1} \frac{\frac{\sqrt{3}}{2}(2k-1)}{(k^2+k+1)^2}\Big) \\ &< 0.9 \end{aligned} \tag{16}$$

and

$$\begin{aligned} \operatorname{Im}(S_1(e^{i\theta})) &> -0.9 - \sum_{k=2}^{a_r-1} \frac{\frac{\sqrt{3}}{2}(2k-1)}{(k^2-3k+3)^2} \\ &> -0.9 - 3.30834 \\ &> -4.20834. \end{aligned} \tag{17}$$

The imaginary part of $S_2(e^{i\theta})$ satisfies

$$\operatorname{Im}(S_2(e^{i\theta})) = (\tilde{w}_{a_r} - \tilde{v}_{b_r}) \sum_{k=1}^{a_r-1} G_{\tilde{w}_k,\tilde{v}_{b_r-a_r+k}}(\theta).$$

For $1 \le k \le a_r - 2$, by Lemma 3.3 (i),

$$G_{\tilde{w}_k,\tilde{v}_{b_r-a_r+k}}(\theta) < G_{\tilde{w}_k,\tilde{v}_{b_r-a_r+k}}(\pi/3) < \frac{\sqrt{3}(k-a_r+\frac{1}{2})}{((a_r-k)^2+a_r-k+1)^2}$$

and

$$G_{\tilde{w}_k,\tilde{v}_{b_r-a_r+k}}(\theta) > G_{\tilde{w}_k,\tilde{v}_{b_r-a_r+k}}(2\pi/3) > \frac{\sqrt{3}(k-a_r+\frac{1}{2})}{((a_r-k)^2-3(a_r-k)+3)^2}.$$

Since $0 < \tilde{w}_{a_r} - \tilde{v}_{b_r} < 1$ and using Lemma 3.3 (iii), we have

$$\begin{aligned} \operatorname{Im}(S_2(e^{i\theta})) &< 0.9 + \sum_{k=1}^{a_r-2} \frac{\sqrt{3}(k-a_r+\frac{1}{2})}{((a_r-k)^2+a_r-k+1)^2} \\ &< 0.9 - \sum_{k=2}^{a_r-1} \frac{\sqrt{3}(k+\frac{1}{2})}{(k^2+k+1)^2} \\ &< 0.9 \end{aligned} \tag{18}$$

and

$$
\begin{aligned}
\mathrm{Im}(S_2(e^{i\theta})) &> (\tilde{w}_{a_r} - \tilde{v}_{b_r})\Big( -0.9 + \sum_{k=1}^{a_r-2} \frac{\sqrt{3}(k - a_r + \frac{1}{2})}{((a_r-k)^2 - 3(a_r-k)+3)^2}\Big) \\
&> (\tilde{w}_{a_r} - \tilde{v}_{b_r})\Big( -0.9 - \sum_{k=2}^{a_r-1} \frac{\sqrt{3}(k + \frac{1}{2})}{(k^2-3k+3)^2}\Big) \\
&> (\tilde{w}_{a_r} - \tilde{v}_{b_r})(-0.9 - 5.28674) \\
&> -6.18674. \qquad (19)
\end{aligned}
$$

The imaginary part of $S_3(e^{i\theta})$ satisfies

$$
\mathrm{Im}(S_3(e^{i\theta})) = \sum_{k=a_r}^{b_r-1} (2k + v_0 - \tilde{v}_{b_r}) G_{v_k, \tilde{v}_{b_r-k}}(\theta).
$$

For $k \geq 2$, by Lemma 3.3 (ii),

$$
G_{v_k,\tilde{v}_{b_r-k}}(\theta) < G_{v_k,\tilde{v}_{b_r-k}}(2\pi/3) < \frac{\sqrt{3}}{(k^2-k+1)(k^2-3k+3)}
$$

and

$$
G_{v_k,\tilde{v}_{b_r-k}}(\theta) > G_{v_k,\tilde{v}_{b_r-k}}(\pi/3) > -\frac{\sqrt{3}}{(k^2-3k+3)(k^2-k+1)}.
$$

Therefore,

$$
\begin{aligned}
\mathrm{Im}(S_3(e^{i\theta})) &< \sum_{k=a_r}^{b_r-1} \frac{\sqrt{3}(2k+v_0-\tilde{v}_{b_r})}{(k^2-k+1)(k^2-3k+3)} \\
&< \frac{30\sqrt{3}}{7} + \sum_{k=a_r+k_1}^{b_r-1+k_1} \frac{2\sqrt{3}}{k(k-1)(k-3)} \\
&< 7.9042 \qquad (20)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Im}(S_3(e^{i\theta})) &> -\sum_{k=a_r}^{b_r-1} \frac{\sqrt{3}(2k+v_0-\tilde{v}_{b_r})}{(k^2-3k+3)(k^2-k+1)} \\
&> -\frac{30\sqrt{3}}{7} - \sum_{k=a_r+k_1}^{b_r-1+k_1} \frac{2\sqrt{3}}{k(k-1)(k-3)} \\
&> -7.9042. \qquad (21)
\end{aligned}
$$

By (7), (16), (19) and (21), we have

$$
\mathrm{Im}(S_{w,v}(e^{i\theta}, r)) < 14.99 \qquad (22)
$$

and by (17), (18) and (20),

$$\operatorname{Im}(S_{w,v}(e^{i\theta}, r)) > -13.01. \tag{23}$$

□

The following corollary of Theorem 3.1 is crucial for the next section.

**Corollary 3.4.** *Let $w = (\overline{a_1, \dots, a_n})$ and $v = (\overline{b_1, \dots, b_m})$ be two purely periodic quadratic numbers. If $b_r \geq M a_r$ for every $r$, then there exist constants $K_1(M)$ and $K_2(M)$ such that*

$$K_2(M) < \cos\theta \operatorname{Im}(S_{w,v}(e^{i\theta}, r)) + \sin\theta \operatorname{Re}(S_{w,v}(e^{i\theta}, r)) < K_1(M).$$

*Moreover if $M \geq e^{55}$, then $K_1(M)$ and $K_2(M)$ are positive.*

*Proof.* This follows easily from the bounds (14), (15), (22), (23) together with the simple observation that for $\pi/3 \leq \theta \leq 2\pi/3$, $-1/2 \leq \cos\theta \leq 1/2$ and $\sqrt{3}/2 \leq \sin\theta \leq 1$. □

## 4. Values of Modular functions

We start this section by looking at the sequence of values $f^{nor}((\overline{N}))$, $N > 2$ for a modular function which is real on the geodesic arc $\{e^{i\theta} \colon \pi/3 \leq \theta \leq 2\pi/3\}$. We have

**Theorem 4.1.** *Let $f(z) = \sum_{n \geq 0} c(n) q^n$ be a modular function which is real valued on the geodesic arc $\{e^{i\theta} \colon \pi/3 \leq \theta \leq 2\pi/3\}$. For any positive integer $N > 2$, the value $f^{nor}((\overline{N}))$ for the quadratic number $w = (\overline{N})$ is real and*

$$\lim_{N \to \infty} f^{nor}((\overline{N})) = -\int_{\pi/3}^{2\pi/3} f(e^{i\theta}) \sin\theta \, d\theta = c(0).$$

*In particular we have*

$$\lim_{N \to \infty} j^{nor}((\overline{N})) = 744.$$

*Proof.* Let $w = (\overline{N})$. We have that $w = \frac{N + \sqrt{N^2 - 4}}{2}$ and $Q_w = [1, -N, 1]$. Hence Pell's equation becomes

$$t^2 - (N^2 - 4)u^2 = 4,$$

with $(N, 1)$ being the smallest positive solution. Thus $\varepsilon_w = \frac{N + \sqrt{N^2 - 4}}{2}$.

Since $w_k = (k, \overline{N}) = -\tilde{w}_{N-k}$,

$$f^{nor}((\overline{N})) = \int_{\rho}^{\rho^2} \frac{f(z)}{2 \log \varepsilon_w} \sum_{k=1}^{N-1} D_N(z, k) dz$$

with

$$D_N(z, k) = \frac{1}{z - w_k} - \frac{1}{z + w_k}.$$

Hence

$$\mathrm{Re}(f^{nor}((\overline{N}))) = -\int_{\pi/3}^{2\pi/3} \frac{f(e^{i\theta})}{2\log\varepsilon_w} \sum_{k=1}^{N-1} \cos\theta \mathrm{Im}(D_N(e^{i\theta},k)) + \sin\theta \mathrm{Re}(D_N(e^{i\theta},k)) d\theta$$

and

$$\mathrm{Im}(f^{nor}((\overline{N}))) = \int_{\pi/3}^{2\pi/3} \frac{f(e^{i\theta})}{2\log\varepsilon_w} \sum_{k=1}^{N-1} \cos\theta \mathrm{Re}(D_N(e^{i\theta},k)) - \sin\theta \mathrm{Im}(D_N(e^{i\theta},k)) d\theta.$$

Now

$$\mathrm{Re}(D_N(e^{i\theta},k)) = \frac{2w_k(\cos^2\theta - w_k^2 - \sin^2\theta)}{((\cos\theta - w_k)^2 + \sin^2\theta)((\cos\theta + w_k)^2 + \sin^2\theta)} \tag{24}$$

and

$$\mathrm{Im}(D_N(e^{i\theta},k)) = \frac{-4w_k \sin\theta\cos\theta}{((\cos\theta - w_k)^2 + \sin^2\theta)((\cos\theta + w_k)^2 + \sin^2\theta)}, \tag{25}$$

so in particular

$$\mathrm{Re}(D_N(e^{i\theta},k)) = \mathrm{Re}(D_N(e^{i(\pi-\theta)},k))$$

and

$$\mathrm{Im}(D_N(e^{i\theta},k)) = -\mathrm{Im}(D_N(e^{i(\pi-\theta)},k)).$$

Therefore

$$\mathrm{Im}(f^{nor}((\overline{N}))) = \int_{\pi/3}^{\pi/2} \frac{f(e^{i\theta})}{2\log\varepsilon_w} \sum_{k=1}^{N-1} \cos\theta \mathrm{Re}(D_N(e^{i\theta},k)) - \sin\theta \mathrm{Im}(D_N(e^{i\theta},k)) d\theta +$$
$$\int_{\pi/3}^{\pi/2} \frac{f(e^{i(\pi-\theta)})}{2\log\varepsilon_w} \sum_{k=1}^{N-1} \cos\theta \mathrm{Re}(D_N(e^{i(\pi-\theta)},k)) - \sin\theta \mathrm{Im}(D_N(e^{i(\pi-\theta)},k)) d\theta$$
$$= 0.$$

Since $w_k = k - \frac{1}{a_{r+1} - \frac{1}{\cdots}}$, it follows from $\varepsilon_w = \frac{N+\sqrt{N^2-4}}{2}$, (24) and (25) that, for all $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$,

$$\lim_{N\to\infty} \frac{1}{2\log\varepsilon_w} \sum_{k=1}^{N-1} \cos\theta \mathrm{Im}(D_N(e^{i\theta},k)) + \sin\theta \mathrm{Re}(D_N(e^{i\theta},k)) = \sin\theta.$$

Thus

$$\lim_{N\to\infty} f^{nor}((\overline{N})) = -\int_{\pi/3}^{2\pi/3} f(e^{i\theta}) \sin\theta \, d\theta = c(0).$$

□

Our next result compares the values of a modular function at two different quadratic irrationalities by comparing their corresponding partial quotients. The results that were given in the introduction will then follow as corollaries of this main result. More precisely we have

**Theorem 4.2.** *Let* $f(z) = \sum_{n\geq 0} c(n)q^n$ *be a modular function which is real valued on the geodesic arc* $\{e^{i\theta}\colon \pi/3 \leq \theta \leq 2\pi/3\}$. *Suppose that*

$$\mathrm{Re}(f^{nor}((1+\sqrt{5})/2)) < c(0). \tag{26}$$

*Then the following holds: Let* $w$ *and* $v$ *be two quadratic numbers with respective periods* $\overline{a_1,\ldots,a_n}$ *and* $\overline{b_1,\ldots,b_m}$ *such that* $m$ *divides* $n$. *If* $M = e^{55}$ *and* $b_r \geq Ma_r$ *for all* $r = 1,\ldots,n$, *then*

$$\mathrm{Re}(f^{nor}(w)) < \mathrm{Re}(f^{nor}(v)).$$

**Remark 4.3.** *As the proof of Theorem 4.2 will show, the condition (26) can be replaced by the condition that* $\mathrm{Re}(f^{nor}(\alpha)) < c(0)$ *for some quadratic number* $\alpha$.

*Proof.* We have that

$$f(w) - f(v) = \int_{\rho}^{\rho^2} f(z) \sum_{r=1}^{n} S_{w,v}(z,r)dz$$

with $S_{w,v}(z,r)$ defined as in (4), so

$$\mathrm{Re}(f(w)-f(v)) = -\int_{\pi/3}^{2\pi/3} f(e^{i\theta}) \sum_{r=1}^{n} \cos\theta \mathrm{Im}(S_{w,v}(e^{i\theta},r)) + \sin\theta \mathrm{Re}(S_{w,v}(e^{i\theta},r))d\theta.$$

By Theorem 3.1 and Corolary 3.4 we obtain

$$\mathrm{Re}(f(v)) = \mathrm{Re}(f(w)) + \mu(M) \tag{27}$$

with

$$nK_2(M)\int_{\pi/3}^{2\pi/3} f(e^{i\theta})d\theta < \mu(M) < nK_1(M)\int_{\pi/3}^{2\pi/3} f(e^{i\theta})d\theta,$$

$K_1(M)$, $K_2(M)$ being the positive constants from Corollary 3.4.

In particular, if $f = 1$, then

$$\log\varepsilon_v = \log\varepsilon_w + \lambda(M) \tag{28}$$

with

$$\frac{\pi}{3}nK_2(M) < \lambda(M) < \frac{\pi}{3}nK_1(M).$$

By definition, the inequality

$$\mathrm{Re}(f^{nor}(w)) < \mathrm{Re}(f^{nor}(v)) \tag{29}$$

holds if and only if

$$\mathrm{Re}(f(w))\log\varepsilon_v < \mathrm{Re}(f(v))\log\varepsilon_w. \tag{30}$$

Now (27) and (28) imply that (30) is equivalent to

$$\mathrm{Re}(f^{nor}(v)) < \frac{\mu(M)}{\lambda(M)} \tag{31}$$

or also to

$$\operatorname{Re}(f^{nor}(w)) < \frac{\mu(M)}{\lambda(M)}. \tag{32}$$

Since the last inequality does not depend on $v$ and is equivalent to (31), the inequality (31) holds either for all or for no $v$.

We first show that $\dfrac{\mu(M)}{\lambda(M)} > c(0)$. Suppose on the contrary $\dfrac{\mu(M)}{\lambda(M)} \leq c(0)$. Let $v = (\overline{N})$. For any $\epsilon > 0$, for large enough $N > N_0$ using Theorem 4.1 we have $\operatorname{Re}(f^{nor}(v)) > c(0) - \epsilon > \frac{\mu(M)}{\lambda(M)}$.

But then, since (31) is equivalent to (29), for $N > \max\{3M, N_0\}$ and $w = \frac{1+\sqrt{5}}{2}$, we have

$$\operatorname{Re}(f^{nor}(w)) = \operatorname{Re}(f^{nor}(1+\sqrt{5})/2) > \operatorname{Re}(f^{nor}(v)) > c(0) - \epsilon.$$

Since this contradicts the assumption (26), we must indeed have $\dfrac{\mu(M)}{\lambda(M)} > c(0)$. Then with $w = \frac{1+\sqrt{5}}{2}$, we have $\operatorname{Re}(f^{nor}(w)) = \operatorname{Re}(f^{nor}(\frac{1+\sqrt{5}}{2})) < c(0) < \dfrac{\mu(M)}{\lambda(M)}$. Hence for every $v = (\overline{b_1, \ldots, b_m})$ with $b_r > 3M$ we have that

$$\operatorname{Re}(f^{nor}(v)) < \frac{\mu(M)}{\lambda(M)}.$$

Now choose $v_0 = (\overline{b_1, \ldots, b_m})$ with $b_r > \max\{3M, Ma_r\}$. Then for this $v_0$ we have $\operatorname{Re}(f^{nor}(v_0)) < \dfrac{\mu(M)}{\lambda(M)}$ and hence $\operatorname{Re}(f^{nor}(w)) < \dfrac{\mu(M)}{\lambda(M)}$. But this equivalent to

$$\operatorname{Re}(f^{nor}(w)) < \operatorname{Re}(f^{nor}(v)).$$

□

We have the following immediate corollaries.

**Corollary 4.4.** *For any quadratic number $w$, and any modular function $f$ which satisfies the conditions of Theorem 4.2 we have*

$$\operatorname{Re}(f^{nor}(w)) \leq -\int_{\pi/3}^{2\pi/3} f(e^{i\theta}) \sin\theta \, d\theta = c(0).$$

*Proof.* For any $w = (\overline{a_1, \ldots, a_n})$, choose $N \in \mathbb{N}$ large enough, so that $N \geq e^{55} a_r$ for all $1 \leq r \leq n$. Then by Theorem 4.2 we have

$$\operatorname{Re}(f^{nor}(w)) < \operatorname{Re}(f^{nor}(\overline{N})).$$

Using Theorem 4.1, this gives

$$\operatorname{Re}(f^{nor}(w)) < \lim_{N\to\infty} \operatorname{Re}(f^{nor}(\overline{N})) = c(0).$$

□

**Corollary 4.5.** *Let $f$ be a modular function which satisfies (26) and $v$ be a quadratic number. If all the partial quotients in the period of $v$ are greater than 3M, then*

$$\mathrm{Re}(f^{nor}(v)) \geq f^{nor}((1+\sqrt{5})/2).$$

*Proof.* This follows from Theorem 4.2 applied to $w = \frac{1+\sqrt{5}}{2} = (2, \bar{3})$. □

Finally, these corollaries prove Theorem 1 since for the $j$ invariant the assumption in the statement of Theorem 4.2 is easily verified. Namely, we have

$$\mathrm{Re}(j^{nor}((1+\sqrt{5})/2)) = 706.3248\ldots < 744.$$

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